# 1 Chapter 3

## 1.1 Fractions

1. Exercise 1. Let $S$ be a multiplicatively closed subset of a ring $A$ and let $M$ be a finitely generated $A$-module. Prove that $S^{-1}M = 0$ if and only if $\operatorname{Ann} M$ meets $S$.

   Exercise 1 solution. We have $\operatorname{Ann} S^{-1}M = S^{-1}\operatorname{Ann} M$ by 3.14. Now $S^{-1}\operatorname{Ann} M$ is the unit ideal in $S^{-1}A$ if and only if $\operatorname{Ann} M$ meets $S$, by 3.11(ii). Meanwhile, $S^{-1}M = 0$ if and only if $\operatorname{Ann} S^{-1}M$ is the unit ideal. This does it.

2. Exercise 2. Let $\mathfrak{a} \lhd A$, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Use this and the Nakayama lemma to give an alternative proof of 2.5, the claim that if $M$ is a finitely generated $A$-module and $\mathfrak{a}M = M$, then there exists $x \equiv 1 \mod \mathfrak{a}$ such that $xM = 0$.

   Exercise 2 solution. Let $a/s$ be an arbitrary element of $S^{-1}\mathfrak{a}$. We have $s = 1 + a'$ for $a, a' \in \mathfrak{a}$, and so $1 + a/s = (1 + a' + a)/(1 + a')$, and this is clearly a unit in $S^{-1}A$ since numerator and denominator are both in $S$. Thus $1 + a/s$ is a unit for all $a/s \in S^{-1}\mathfrak{a}$, and since this is an ideal it follows that $1 + (x)(a/s)$ is a unit for all $x \in S^{-1}A$. Thus, $a/s$ is in the Jacobson radical, by 1.9. This proves that $S^{-1}\mathfrak{a} \subset \mathfrak{J}_{S^{-1}A}$.

   Now let $M$ be finitely generated and let $\mathfrak{a} \lhd A$ be such that $\mathfrak{a}M = M$. Let $S = 1 + \mathfrak{a}$ as we've done here. Then $S^{-1}\mathfrak{a}$ is in the Jacobson radical of $S^{-1}A$. But also, $S^{-1}\mathfrak{a}S^{-1}M = S^{-1}M$, because $\mathfrak{a}M \supset M$. Also, $S^{-1}M$ is finitely generated over $S^{-1}A$, by the images of the generators of $M$ under the map $m \mapsto m/1$. So by Nakayama, $S^{-1}M = 0$. Then by problem 1, $\operatorname{Ann} M$ meets $S$. I.e., there exists $x \in \operatorname{Ann} M$ with $x \equiv 1 \mod \mathfrak{a}$. This does it.

   Now for completeness, I note that as I have things set up, this proof of 2.5 is really circular, because the proof of Nakayama that I recorded depends on 2.5. There is a proof given by A and M that doesn't, which I declined to write down at the time, but here it is now. Let $M$ be finitely generated and nonzero, and let $x_1, \dots, x_n$ be a minimal set of generators for $M$. If $\mathfrak{a}$ is such that $\mathfrak{a}M = M$, then $x_n$ has the form

   $$a_1x_1 + \cdots + a_nx_n$$

   for $a_1, \dots, a_n \in \mathfrak{a}$, so that

   $$(1 - a_n)x_n = a_1x_1 + \cdots + a_{n-1}x_{n-1}$$

   If $\mathfrak{a} \subset \mathfrak{J}$, then $1 - a_n$ is a unit. This implies that $x_n \in \langle x_1, \dots, x_{n-1}\rangle$, contradicting minimality of the set of generators. This proves it.

3. Exercise 3. Let $A$ be a ring and let $S,T$ be two different multiplicatively closed subsets. Let $U$ be the image of $T$ under $A \to S^{-1}A$. Show that $(ST)^{-1}A$ is isomorphic to $U^{-1}(S^{-1}A)$.

   Exercise 3 solution. We have a map $A \xrightarrow{\phi} S^{-1}A \xrightarrow{\psi} U^{-1}(S^{-1}A)$. Clearly everything in $ST$ ends up a unit under the composed map. Also, to be in the kernel of $\psi$ you have to annihilate an element of $U$; thus if $x$ is in the kernel of $\psi\circ\phi$ we have $\phi(x)u = 0$ in $S^{-1}A$, so that $\phi(xt) = 0$ in $S^{-1}A$ for some $t$ such that $u = \phi(t)$. But if $\phi(xt) = 0$ in $S^{-1}A$, then $xt$ is annihilated by an element of $s$ in $A$. So $x$ is annihilated by $st \in ST$. Thus $\psi\circ\phi(x) = 0$ implies $\exists st \in ST$ with $stx = 0$ in $A$. Lastly, everything in $S^{-1}A$ has the from $x/s = \phi(x)\phi(s)^{-1}$, so everything in $U^{-1}(S^{-1}A)$ has the form $\phi(x)\phi(s)^{-1}/u = \phi(x)\phi(s)^{-1}/\phi(t) = \psi\circ\phi(x)\psi\circ\phi(s)^{-1}\psi\circ\phi(t)^{-1} = \psi\circ\phi(x)\psi\circ\phi(st)^{-1}$. Thus the homomorphism $\psi\circ\phi$ and the ring $U^{-1}(S^{-1}A)$ satisfy the conditions that uniquely characterize the ring $(ST)^{-1}A$ and its canonical homomorphism from $A$. So they are isomorphic.

4. Exercise 4. Let $f : A \to B$ be a homomorphism of rings, let $S \subset A$ be a multiplicative submonoid of $A$, and let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

   Exercise 4 solution. Okay $T^{-1}B$ is the localization of $B$ as a ring, while $S^{-1}B$ is the localization of $B$ as an $A$-module. Thus elements of $T^{-1}B$ look like $b/f(s)$ while elements of $S^{-1}B$ look like $b/s$. The former are multiplied by elements of $S^{-1}A$ by $(a/s)(b/f(s')) = f(a)b/f(ss')$, while the latter by $(a/s)(b/s') = f(a)b/ss'$.

   There is an evident map $\phi : S^{-1}B \to T^{-1}B$ defined by $b/s \mapsto b/f(s)$. The fact that $f$ is a ring homomorphism makes this an $S^{-1}A$-module homomorphism:

$$\begin{aligned}\phi((a/s)(b/s')) &= \phi(f(a)b/ss')\\ &= f(a)b/f(s)f(s')\\ &= (a/s)(b/f(s')) = (a/s)\phi(b/s')\end{aligned}$$

   and

$$\begin{aligned}\phi(b/s + b'/s') &= \phi((f(s')b + f(s)b')/ss')\\ &= (f(s')b + f(s)b')/f(s)f(s')\\ &= b/f(s) + b'/f(s')\\ &= \phi(b/s) + \phi(b/s')\end{aligned}$$

   We need to show it is an isomorphism. It is obviously surjective, since $S$ surjects onto $T = f(S)$ by definition. Meanwhile, we have $b/f(s) = 0/1$ if $b$ is annihilated by some element $f(t)$ of $T$. But this is what it means for $b$ (as an element of the $A$-module

$B$) to be eliminated by $t$; so $b/f(s) = 0$ in $T^{-1}B$ implies $b/s = 0$ in $S^{-1}B$. So the map is injective too.

I have something I'd like to add, for my later thinking. $S^{-1}B$ gets the structure of an $S^{-1}A$-algebra (not just a module) by defining multiplication in the obvious way by $(b/s)(b'/s') = bb'/ss'$. On this definition, $\phi : S^{-1}B \to T^{-1}B$ is actually an $S^{-1}A$-*algebra* isomorphism, not just a module isomorphism, since $\phi(b/s)\phi(b'/s') = (b/f(s))(b'/f(s')) = bb'/f(s)f(s') = bb'/f(ss') = \phi(bb'/ss') = \phi((b/s)(b'/s'))$.

5. Exercise 8(i)-(iii). Suppose $S, T \subset A$ are multiplicatively closed and $S \subset T$, so there is a mapping $\phi : S^{-1}A \to T^{-1}A$ mapping $a/s \mapsto a/s$. Show the following conditions on $S, T, \phi$ are equivalent:

   (a) $\phi$ bijective.

   (b) $\forall t \in T$, $t/1$ is a unit in $S^{-1}A$.

   (c) $\forall t \in T$, $\exists x \in A$ s.t. $xt \in S$.

   Exercise 8(i)-(iii) solution. It is clear $\phi$ is a ring homomorphism.

   (a)$\Rightarrow$(b). It's a bijection so an isomorphism, and $t/1$ is a unit in $T^{-1}A$. Check.

   (b)$\Rightarrow$(c). $t/1$ a unit in $S^{-1}A$. Then $\exists a/s$ such that $at/s = 1/1$. So $\exists u \in S$ such that $uat = us$. Take $x = ua$. Then $xt = us \in S$.

   (c)$\Rightarrow$(a). We will first show $\phi$ is surjective. For all $x/t \in T^{-1}A$, we need to show there exists $y/s \in S^{-1}A$ with $\phi(y/s) = x/t$. Now by assumption there is some $z \in A$ with $zt \in S$. Then let $y = zx$ and $s = zt$. Clearly we have $\phi(y/s) = zx/zt = x/t$ in $T^{-1}A$, so $\phi$ is surjective.

   Now we will show $\phi$ is injective. We must prove that if $x/s = 0$ in $T^{-1}A$, it also $= 0$ in $S^{-1}A$. But if $x/s = 0$ in $T^{-1}A$, there is a $t \in T$ with $tx = 0$ in $A$. By assumption, there is a $y \in A$ with $yt \in S$. Then $(yt)x = 0$, which implies that $x/s = 0$ in $S^{-1}A$. Thus $\phi$ is injective.

## 1.2 Spec and presheaves

1. Exercise 20. Let $f : A \to B$ be a ring homomorphism and $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ its pullback. Show that

   (a) $f^*$ is surjective if and only if every prime ideal of $A$ is contracted.

   (b) $f^*$ is injective if every prime ideal of $B$ is extended.

   Is the converse to (b) also true?

Exercise 20 solution. (a) is an immediate consequence of proposition 3.16. To say $f^*$ is surjective is to say every prime ideal of $A$ is contracted from a prime; it was shown in 3.16 that a prime ideal is contracted from a prime if and only if it is contracted period.

(b) To say $f^*$ is injective is to say that no two prime ideals of $B$ contract to the same ideal of $A$. If every prime ideal of $B$ is extended, then every prime ideal is the extension of its contraction (see ch. 1, proposition 1.17). This implies no two distinct prime ideals can contract to the same ideal, or they would be equal to this ideal's extension and thus each other.

In the other direction, we *can* have an injective map $f^*$ without every prime of $B$ being extended, so the converse is false. For example, let $A = k$, a field, and let $B = k[x]/(x^2)$, and let $f$ be the obvious inclusion. Then the only prime ideal of $B$ is $(x)$, because $B$ is actually a local ring: $a + bx \in B$ is a unit unless $a = 0$. So the map $f^*$ is injective. However, $(x)$ is not an extended ideal because its contraction to $A$ is $(0)$, and this extends to $(0)$.

2. Exercise 21.

    (a) Let $A$ be a ring, $S$ a multiplicatively closed subset of $A$, and $\phi : A \to S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}\,A$ is a homeomorphism to its image. Let $X = \mathrm{Spec}\,A$ and $S^{-1}X = \mathrm{im}\,\phi^*$. In particular, show that if $f \in A$, the image of $\mathrm{Spec}\,A_f$ in $X$ is the basic open set $X_f$ defined in Ch. 1, ex. 17.

    Exercise 21a solution. This is actually kind of obvious I think. We know from proposition 3.11(iv) that $\phi^*$ is a bijection onto its image. Since $\phi^{-1}$ respects containment of ideals, and the topology on Spec is defined in terms of ideal containment, this pretty much makes it a homeomorphism. I suppose we could talk through this conclusion, since although $J \supset I \Rightarrow \phi^{-1}(J) \supset \phi^{-1}I$, $J \not\supset I$ does not imply $\phi^{-1}(J) \not\supset \phi^{-1}(I)$.

    We know $\phi^*$ is continuous from Ch. 1 exercise 21. We need to check $(\phi^*)^{-1}$ is continuous. That is, we need to check $\phi$ pushes closed sets of prime ideals in $\mathrm{im}\,f^*$ forward to closed sets of prime ideals in $\mathrm{Spec}\,S^{-1}A$. $\mathrm{im}\,f^*$ is precisely the set of contracted prime ideals in $A$. A closed subset is the set of contracted prime ideals that contain some ideal $\mathfrak{a}$ of $A$. A contracted prime ideal $\mathfrak{p} \lhd A$ pushes forward to $S^{-1}\mathfrak{p}$, which is also prime. (Quick proof: $A/\mathfrak{p}$ is a domain and $\mathfrak{p}$ does not meet $S \Rightarrow S^{-1}(A/\mathfrak{p}) = (S^{-1}A/S^{-1}\mathfrak{p})$ is contained in $\mathrm{Frac}(A/\mathfrak{p})$ and is thus a domain.) So what needs to be shown is that $\mathfrak{p}$ contains $\mathfrak{a}$ and doesn't meet $S$ if and only if $S^{-1}\mathfrak{p}$ contains $S^{-1}\mathfrak{a}$ and isn't the unit ideal. But this is clear: if $\mathfrak{p}$ contains $\mathfrak{a}$, then $S^{-1}\mathfrak{p}$ contains $S^{-1}\mathfrak{a}$; this much is obvious. Conversely, if $S^{-1}\mathfrak{p}$ contains $S^{-1}\mathfrak{a}$, then for all $a/s \in S^{-1}\mathfrak{a}$ we have $p/s' \in S^{-1}\mathfrak{p}$

such that $p/s' = a/s$, so that $sp - s'a$ is annihilated by $t \in S$: $tsp = ts'a$. Then $ts'a \in \mathfrak{p}$, so that either $t$, $s'$, or $a$ is in $\mathfrak{p}$ since the latter is prime. Since we are presuming that $S$ does not meet $\mathfrak{p}$, it must be $a \in \mathfrak{p}$. This shows $\mathfrak{a} \subset \mathfrak{p}$ and proves that $\phi^*$ is a homeomorphism to its image. This justifies the use of the symbol $S^{-1}X$ to refer to this image in $X$.

In particular, if $f \in A$, $S = \{1, f, f^2, \dots\}$, then $\operatorname{Spec} S^{-1}A$ is the set of all prime ideals that do not contain $f$ or any power. Because they are prime, these are exactly the ones that don't contain $f$. This is $X_f$ defined in Ch. 1, ex. 17. Davesh Maulik (or Mumford I think!) would call it $D(f)$.

(b) Let $f : A \to B$ be a ring homomorphism and let $Y = \operatorname{Spec} B$. Let $f^* : Y \to X$ be the mapping associated with $f$. Identifying $\operatorname{Spec} S^{-1}A$ with its canonical image $S^{-1}X$ in $X$, and similarly $\operatorname{Spec} S^{-1}B$ with $S^{-1}Y$, show that $S^{-1}f^* : \operatorname{Spec} S^{-1}B \to S^{-1}A$ is the restriction of $f^*$ to $S^{-1}Y$, and that $S^{-1}Y = (f^*)^{-1}(S^{-1}X)$.

Exercise 21b solution. This is just a statement about sets and no longer about topology, since we are allowed to view $S^{-1}X$ as a subset of $X$ by the last problem. (For clarity of thinking I note that $S^{-1}B \cong f(S)^{-1}B$ as rings; see problem 4.)

What is $S^{-1}f^*$? Well, what is $S^{-1}f$? This is the map $a/s \mapsto f(a)/f(s)$. $S^{-1}f^*$ is the way prime ideals pull back under this map. We need to show that a prime ideal in $S^{-1}B$ pulls back to a prime ideal in $S^{-1}A$ if and only if the corresponding ideal in $B$ pulls back to the corresponding ideal in $A$ under $f$. Say $f(S)^{-1}\mathfrak{q}$ pulls back to $S^{-1}\mathfrak{p}$. Then for all $p/s \in S^{-1}\mathfrak{p}$, there must exist ... too many details. Take a Yash-like birds-eye view:

Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow_{\pi} & & \downarrow_{\pi'} \\ S^{-1}A & \xrightarrow{S^{-1}f} & S^{-1}B \end{array}$$

The pullbacks are

$$\begin{array}{ccc} \operatorname{Spec} A & \xleftarrow{f^*} & \operatorname{Spec} B \\ \cup & & \cup \\ \operatorname{Spec} S^{-1}A & \xleftarrow{S^{-1}f^*} & \operatorname{Spec} S^{-1}B \end{array}$$

The statement that $S^{-1}f^* = f^*|_{\operatorname{Spec} S^{-1}B}$ is just the statement that the bottom diagram commutes. This is immediate from the functoriality of the pullback, which we proved in Ch. 1, problem 21! I suppose good form would be to check that the top diagram commutes, but this is actually clear from the definitions of the maps.

Meanwhile, once we have this, the statement that $(f^*)^{-1}(S^{-1}X) = S^{-1}Y$ is the statement that the only prime ideals in $B$ that pull back to prime ideals of $A$ not meeting $S$ are those that don't meet $f(S)$. In other words, that if a prime ideal of $B$ meets $f(S)$, its pullback in $A$ meets $S$. This is totally obvious.

(c) Let $\mathfrak{a} \lhd A$ and let $\mathfrak{b} = \mathfrak{a}^e$. Let $\bar{f} : A/\mathfrak{a} \to B/\mathfrak{b}$. If $\operatorname{Spec} A/\mathfrak{a}$ is identified with its canonical image $V(\mathfrak{a})$ in $X$ and same for $B, \mathfrak{b}$, show that $\bar{f}^*$ is the restriction of $f^*$ to $V(\mathfrak{b})$.

Exercise 21c solution. This time, the diagram is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/\mathfrak{a} & \xrightarrow{\bar{f}} & B/\mathfrak{b} \end{array}$$

The pullback diagram is

$$\begin{array}{ccc} \operatorname{Spec} A & \xleftarrow{f^*} & \operatorname{Spec} B \\ \cup & & \cup \\ V(\mathfrak{a}) & \xleftarrow{\bar{f}^*} & V(\mathfrak{b}) \end{array}$$

and again, the result follows from the functoriality of the pullback.

For use in the next problem: do we also have $(\bar{f}^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{b})$? In other words, if a prime ideal pulls back to one containing $\mathfrak{a}$, does it have to contain $\mathfrak{b}$? Yes; NOT DONE HERE.

(d) Let $\mathfrak{p}$ be a prime ideal of $A$. Take $S = A \smallsetminus \mathfrak{p}$ and reduce $S^{-1}A$ mod $S^{-1}\mathfrak{p}$. Deduce that $(f^*)^{-1}(\mathfrak{p})$ is naturally homeomorphic to $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$. $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the *fiber* of $f^*$ over $\mathfrak{p}$.

Exercise 21d solution. This combines the last two problems. The diagram looks like

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_\mathfrak{p} & \xrightarrow{f_\mathfrak{p}} & B_\mathfrak{p} \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) & \xrightarrow{\bar{f}_\mathfrak{p}} & B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}B_\mathfrak{p} \end{array}$$

By the above discussion, the pullback diagram is commutative:

$$\begin{array}{ccc}
\operatorname{Spec} A & \xleftarrow{f^*} & \operatorname{Spec} B \\
\cup & & \cup \\
\operatorname{Spec} A_{\mathfrak{p}} & \xleftarrow{f_{\mathfrak{p}}^*} & \operatorname{Spec} B_{\mathfrak{p}} \\
\cup & & \cup \\
\operatorname{Spec} k(\mathfrak{p}) & \xleftarrow{\bar{f}_{\mathfrak{p}}^*} & \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}})
\end{array}$$

$k(\mathfrak{p})$ has the lone prime ideal $(0)$. Following this ideal up the left side and pulling back across the top, we see that this is $(f^*)^{-1}(\mathfrak{p})$ in $\operatorname{Spec} B$. Pulling back across the bottom, we see that it is $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}})$ which is then embedded in $\operatorname{Spec} B$ by the above discussion. So we conclude that these are homeomorphic. Let's also just check that

$$B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} = k(\mathfrak{p}) \otimes_A B$$

to complete the argument. We have $B_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A B$ by proposition 3.5. Then, we have

$$B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}B_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A B) = k(\mathfrak{p}) \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A B) = k(\mathfrak{p}) \otimes_A B$$

by exercise 2 of chapter 2. This finishes it!

3. Exercise 23. Let $A$ be a ring, $X = \operatorname{Spec} A$, and let $U$ be a basis open for $X$, i.e. $U = D(f)$ for some $f \in A$.

   (a) Show that $A(U)$, defined as $A_f$ for $U = D(f)$, doesn't depend on $f$ but only on $U$!

   Exercise 23a solution. Cool! Geometrically this is saying that the ring of regular functions on the complement of a hypersurface doesn't depend on the function that was used to cut out the hypersurface. For example, $A_f$ and $A_{f^2}$ are the same. (I remember in the case of varieties, it was actually a theorem, not a definition, that the ring of regular functions on $D(f)$ was $k[X]_f$.)

   Let's see. Suppose $f$ and $g$ are such that a prime ideal contains $f$ if and only if it contains $g$. We want to say that $A_f$ and $A_g$ are isomorphic.

   The problem is custom-made for the universal property of localizations. If $S$ is a multiplicatively closed set in $A$ and $\phi : A \to B$ is a homomorphism such that $\phi(S) \subset B^*$, then $\phi$ factors through $S^{-1}A$. If furthermore, all elements of $B$ have the form $\phi(a)\phi(s)^{-1}$ for some $s \in S$, then the map from $S^{-1}A$ is surjective, and if the only stuff in the kernel of $\phi$ is stuff in $A$ that's annihilated by some $s \in S$, then $S^{-1}A \cong B$. So with $f, g$ as above, first we need to check all powers of $f$ map to units in $A_g$. What is the map?

The map is the canonical homomorphism $\phi : A \to A_g$. The image of $f$ is $f/1$. We need to check this is a unit of $A_g$. (It will follow immediately that all its powers are.)

We know $f, g \in A$ are contained in the same prime ideals. This implies immediately that $(f), (g)$ have the same radical. Then in particular $g \in \sqrt{(f)}$, so that $g^k = af$ for some $k \in \mathbb{N}, a \in A$. It follows that $(f/1)(a/g^k) = 1/1$ in $A_g$ and thus that $f/1$ is a unit.

Secondly, we need to check that everything in $A_g$ has the form $\phi(b)\phi(f^k)^{-1}$ for $b \in A, k \in \mathbb{N} \cup \{0\}$. The key is that because $f \in \sqrt{(g)}$, we have $f^{k'} = a'g$ in $A$, and therefore $\phi(a')\phi(f^{k'})^{-1} = (a'/1)(a/g^k)^{k'}$. This is

$$\frac{a'a^{k'}}{g^{kk'}}$$

I claim this equals $1/g$. Indeed, this is true iff the equation

$$a'a^{k'}g = g^{kk'}$$

holds in $A$. But $a'g = f^{k'}$, and $g^k = af$ so this simplifies to

$$f^{k'}a^{k'} = (af)^{k'}$$

and this is clear. Therefore $1/g$ has the requisite form. As all elements $a/1$ have the requisite form (they $= \phi(a)\phi(1)^{-1}$), and these with $1/g$ generate $A_g$, this shows that all its elements have the requisite form.

Last, we need to check that $\phi(b) = 0$ only if $b$ was already zero in $A_f$, i.e. $b$ is annihilated by some power of $f$. Indeed, if $\phi(b) = 0$ then this means $b$ is annihilated by some power of $g$, say $m$. But then $(a'g)^m b = (f^{k'})^m b = 0$ too, which means $b$ was already annihilated by a power of $f$.

This shows that $A_f \cong A_g$. Thus $A(U)$ depends only on $U = D(f)$ and not the representative $f \in A$ that "cuts out its complement."

(b) Show that if $U, U'$ are two basis opens with $U = D(f), U' = D(g)$ and $U' \subset U$, there exists a "restriction homomorphism" $A(U) \to A(U')$ that only depends on $U, U'$ and not on $f, g$.

Exercise 23b solution. $U' \subset U$ means that the set of prime ideals not containing $g$ is inside the set not containing $f$. Taking complements in $X$, $V(f) \subset V(g)$: every prime containing $f$ contains $g$. This statement is equivalent to $g \in \sqrt{(f)}$: for another way to say it is that $g$ is in the intersection of the primes containing $f$. Thus $\exists k \in \mathbb{N}, a \in A$ such that $g^k = af$. Now define $A(U) = A_f \to A(U') = A_g$ by sending $1/f \mapsto a/g^k$ (and the rest in the obvious way: for $b \in A$ send $b/1 \mapsto b/1$; etc.). This is a homomorphism because $(1/1) = (f/1)(1/f) \mapsto (f/1)(a/g^k) = af/g^k = 1/1$.

We need to show this homomorphism does not depend on $f, g$ (or for that matter on $k, a$) but only on $U, U'$. Thus, let $f, \hat{f}$ be any elements of $A$ such that $U = D(f) = D(\hat{f})$. Let $g, \hat{g}$ be any elements of $A$ such that $U' = D(g) = D(\hat{g})$. By the arguments above, we have isomorphisms $A_f \leftrightarrow A_{\hat{f}}$ and $A_g \leftrightarrow A_{\hat{g}}$ constructed as in part (a), and homomorphisms $A_f \to A_g$ and $A_{\hat{f}} \to A_{\hat{g}}$ constructed as in the last paragraph. What we have to do is show that the diagram

$$\begin{array}{ccc} A_f & \leftrightarrow & A_{\hat{f}} \\ \downarrow & & \downarrow \\ A_g & \leftrightarrow & A_{\hat{g}} \end{array}$$

always commutes. This will show that the down arrows are the same up to conjugation by the across arrows.

I believe that this result will follow from the desired result of part (d), that the homomorphisms constructed here compose properly. For if they do, then the compositions around the square above are equal to the diagonals and thus don't depend on the way around the square. Let us do part (c) then.

(c) Show that if $U = U'$, this homomorphism is the identity.

Exercise 23c solution. Actually, this follows from the way I did part (a), because I constructed the restriction homomorphism in that case, and showed it was an isomorphism.

(d) Show that if $U, U', U''$ are basis opens with $U \supset U' \supset U''$, then

$$\begin{array}{ccc} A(U) & \to & A(U') \\ & \searrow & \downarrow \\ & & A(U'') \end{array}$$

commutes.

Okay, let $U = D(f), U' = D(f'), U'' = D(f'')$. From the inclusions $U \supset U' \supset U''$, we have (taking complements) $V(f) \subset V(f') \subset V(f'')$. Then all the primes containing $f$ also contain $f'$, etc., so $\sqrt{(f)} \supset \sqrt{(f')} \supset \sqrt{(f'')}$. There exists $a \in A, m \in \mathbb{N}$ such that $(f'')^m = af$. There also exist $b, b' \in A$ and $n, n' \in \mathbb{N}$ so that $(f')^n = bf$ and $(f'')^{n'} = b'f'$. We get a map $\phi : A_f \to A_{f''}$ by mapping $1/f \mapsto a/(f'')^m$. We get maps $\psi : A_f \to A_{f'}$ and $\psi' : A_{f'} \to A_{f''}$ by $1/f \mapsto b/(f')^n$ and $1/f' \mapsto b'/(f'')^{n'}$. Our goal is to show $\phi = \psi' \circ \psi$.

It is clear that the maps coincide on the image of $A$ in $A_f$, so the only thing to check is that they do the same thing with $1/f$. On $\psi' \circ \psi$ we have

$$1/f \mapsto b/(f')^n \mapsto b(b')^n/(f'')^{nn'}$$

thus what we have to show is that

$$\frac{a}{(f'')^m} = \frac{b(b')^n}{(f'')^{nn'}}$$

in $A_{f''}$, in other words that

$$a(f'')^{nn'} - b(b')^n(f'')^m$$

is annihilated by some power of $f''$ in $A$. Actually, it is already zero in $A$: using $af = (f'')^m, bf = (f')^n, b'f' = (f'')^{n'}$, we have

$$\begin{aligned} a(f'')^{nn'} - b(b')^n(f'')^m &= a(b'f')^n - ab(b')^n f \\ &= a(b')^n \left[(f')^n - bf\right] = 0 \end{aligned}$$

This completes the proof!

(e) Let $x = \mathfrak{p} \in X$. Show that

$$\varinjlim_{U \ni x} A(U) \cong A_{\mathfrak{p}}$$

Exercise 23d solution. Let's use the universal property of direct limits. Let $R$ be any ring with a homomorphism from every $A(U)$ with $U \ni x$, i.e. $U = D(f)$ with $f \notin \mathfrak{p}$, that commutes with the restriction maps defined above. In other words, $R$ has a map from every $A_f$, $f \notin \mathfrak{p}$, that commutes with the restrictions. Then really what $R$ has is a map from $A$ such that for all $f \notin \mathfrak{p}$, the image of $f$ is a unit. Therefore, $R$ has a unique map from $A_{\mathfrak{p}}$ (by the universal property of localizations)! So $A_{\mathfrak{p}}$ is the desired direct limit.

As A and M note, parts (a)-(d) show that the rings $A(U)$ and restriction maps constitute a presheaf on $X$ (via the specification of a presheaf on the basis opens), and part (e) shows the stalk of this presheaf at $\mathfrak{p} = x$ is $A_{\mathfrak{p}}$.

4. Exercise 24. Show that the presheaf of the last exercise actually gives us a sheaf! In particular, show that if $(U_i)_{i\in I}$ is a cover of $X$ by basis opens and for each $U_i$ there is an $s_i \in A(U_i)$ such that for every pair $i, j$ the images of $s_i$ and $s_j$ in $A(U_i \cap U_j)$ are equal, then there exists a unique $s \in A = A(X)$ whose image in each $A(U_i)$ is $s_i$. (I note that this result can then be applied with $X$ = each basis open $U$ to show that the presheaf has the sheaf property on basis opens, and this in turn implies that it induces a unique sheaf on $X$.)

Exercise 24 solution. Let $U_i = D(f_i)$. First, I claim $U_i \cap U_j = D(f_i f_j)$. Indeed, a prime ideal fails to contain $f_i f_j$ if and only if it fails to contain both $f_i$ and $f_j$. (As an ideal, if it contained either, it would contain the product, and because it is prime, if it contains the product, it contains at least one of them.) Thus $A(U_i \cap U_j) = A_{f_i f_j}$.

Now, suppose that there are $U_i = D(f_i)$ that cover $X = \operatorname{Spec} A$. Then $(\{f_i\}_i) = (1) \lhd A$. This implies that there is a finite linear combination

$$1 = \sum_1^m c_i f_i$$

taking place in $A$. Not only that, but this means $(f_1, \dots, f_m) = (1)$ and therefore $(f_1^n, \dots, f_m^n) = (1)$ for any $n$, so there is a similar equation with $f_i^n$ replacing $f_i$, with the $c_i$ depending on $n$.

Now suppose we have $s_i \in A(U_i) = A_{f_i}$ such that for each $i, j$, the images of $s_i, s_j$ in $A(U_i \cap U_j) = A_{f_i f_j}$ coincide. Represent each $s_i$ as $a_i / f_i^{n_i}$. Then for each pair $i, j$, we have

$$s_i| - s_j| = \frac{a_i f_j^{n_i}}{(f_i f_j)^{n_i}} - \frac{a_j f_i^{n_j}}{(f_i f_j)^{n_j}} = 0$$

in $A_{f_i f_j}$. In other words, in $A$,

$$a_i f_i^{n_j} f_j^{n_i + n_j} - a_j f_i^{n_i + n_j} f_j^{n_i}$$

is annihilated by some power of $f_i f_j$.

For the finite list $s_1, \dots, s_m$, we may actually choose the representations $s_i = a_i / f_i^n$ to all have a uniform exponent in the denominator. Then, for $1 \le i, j \le m$, this last condition can be written more simply:

$$a_i f_j^n - a_j f_i^n$$

is annihilated by a power of $f_i f_j$. Suppose it is $(f_i f_j)^\ell$. Then

$$a_i f_i^\ell f_j^{\ell+n} = a_j f_j^\ell f_i^{\ell+n}$$

Since $s_i = a_i f_i^\ell / f_i^{\ell+n}$ in $A_{f_i}$ and $s_j = a_j f_j^\ell / f_j^{\ell+n}$, this shows, by replacing $a_i f_i^\ell$ with $a_i$ and $\ell + n$ with $n$, that for each of the finite number of pairs $1 \le i, j \le m$, we can choose the form $s_i = a_i / f_i^n$ with big enough $n$ so that we actually have $f_j^n a_i = f_i^n a_j$. Nothing is lost if we go even bigger, so we can choose these representations so that this equation holds for every pair $1 \le i, j \le m$. Then, with $c_i$ chosen to satisfy $1 = \sum c_i f_i^n$ as above, let

$$s = \sum_1^m c_i a_i$$

I claim that $s$ restricts to $s_i$ on each $U_i$, at least those with $1 \leq i \leq m$. Indeed, we have $s|_{U_i} = s/1 \in A_{f_i}$, and then

$$f_i^n s = \sum_1^m c_j f_i^n a_j = \sum_1^m c_j f_n^j a_i = a_i \sum_1^m c_j f_n^j = a_i$$

in $A$, and it follows that $s/1 = a_i/f_i^n = s_i$ in $A_{f_i}$.

But beyond this, I also claim that the $s$ given restricts to $s_i$ on every $U_i$, not just on those in the finite subcover $U_1, \dots, U_m$. I will approach this claim in an oblique way. What I have shown above is two things: (1) given a cover $X = \bigcup U_i$ with each $U_i = D(f_i)$, there is a finite subcover (i.e. $X$ is compact, or quasicompact in the parlance of algebraic geometry); and (2) given a *finite* cover $X = \bigcup_1^m U_i$, and some $s_i \in A(U_i)$ such that for all pairs $i, j$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists an $s$ such that $s|_{U_i} = s_i$ for all $i$. I will now show that the $s$ in (2) is unique with this property. This will allow us to conclude that the $s$ determined from $s_1, \dots, s_m$ will also coincide with all other $s_i$'s: indeed, for any specific $i \notin [m]$, both $U_1, \dots, U_m$ and $U_1, \dots, U_m, U_i$ are finite covers of $X$ by basis opens, and both $s_1, \dots, s_m$ and also $s_1, \dots, s_m, s_i$ have the property of agreeing on overlaps. Thus by (2), they each determine an $s \in A$ (say $s$ and $s'$) that restrict correctly. But then $s'$ restricts correctly on $s$'s $U$'s, thus it equals $s$ by uniqueness. Thus $s$ already agreed with $s_i$ when restricted to $U_i$.

To show uniqueness in (2), because everything is contained in a system of rings so that we can take differences, it is only necessary to show that if all the $s_i$'s are zero, then any $s \in A$ that restricts correctly is also zero. So, suppose we have an $s \in A$ and $f_1, \dots, f_m \in A$ such that the image of $s$ is zero in $A_{f_i}$ for all $i \in [m]$. Let $M = (s) \lhd A$. Then $M$ is an $A$-module and it suffices to show it is zero as an $A$-module. But as we have repeatedly seen, being zero is a very local property. $M$ is zero if $M_\mathfrak{m}$ is zero for all maximals $\mathfrak{m} \lhd A$. But because the $U_i = D(f_i) = \operatorname{Spec} A_{f_i}$ cover $X$, every maximal of $A$ extends to a maximal in $A_{f_i}$ for some $i$. For given $\mathfrak{m}$, we know $s$'s image in the $A_{f_i}$ (such that $D(f_i) \ni \mathfrak{m}$) is zero; then it is zero in $A_\mathfrak{m} = (A_{f_i})_{\mathfrak{m}_{f_i}}$ too, and then the ideal it generates is also zero. Thus $M_\mathfrak{m} = 0$ for all $\mathfrak{m}$, and it follows $M = 0$, thus $s = 0$. We are done.